Exercise 5.x: Trams

This exercise shows formally that the conclusions of the Trams example are very sensitive to an arbitrary prior upper bound M, which was not apparent from a single WinBUGS run. This problem can be alleviated by using the Jeffreys prior.

a) Suppose we bound the uniform distribution above by M, say 5000. What is the posterior distribution?

b) What is the posterior mean $E[N|y]$?

c) Calculate an approximation to $\sum_{N=y}^{M} \frac{1}{N}$, by assuming N is continuous and approximating the sum by an integral. Thus obtain a simplified expression for the posterior mean.

d) Show that the posterior mean increases as $M/\log M$, thus we can make it as big as we want by increasing M.

e) Under the Jeffreys prior, show that as M goes to infinity the posterior median tends to a fixed quantity. First derive the posterior distribution, and then solve $p(N \leq n|y) = 0.5$ for n.

Solution

a) The posterior distribution is $p(N|y) \propto 1/N$; $N = y, y+1, ..., M$. Now $\sum_{N=y}^{M} p(N|y) = 1$, and so $p(N|y) = c/N$, where $c = 1/\sum_{N=y}^{M} \frac{1}{N}$.

b) The posterior mean is then $\sum_{N=y}^{M} Np(N|y) = \sum_{N=y}^{M} Nc/N = \sum_{N=y}^{M} c = c(M - y + 1) = (M - y + 1)/\sum_{N=y}^{M} \frac{1}{N}$.

c) $c^{-1} = \int_y^M \frac{1}{N} dN = \log(M/y)$

d) $E[N|y] \times \frac{\log M}{M} = \frac{M-y+1}{c^{-1}} \frac{\log M}{M} \rightarrow \frac{M-y+1}{M} \frac{\log M}{\log M - \log y} \rightarrow 1$ as M increases.

e) If $M = \infty$, then the posterior distribution $p(N|y) \propto 1/N^2$, $N \geq y$, and so $p(N|y) = c/N^2$, $N \geq y$, where $c^{-1} = \int_y^\infty N^{-2} dN = [-N^{-1}]_y^\infty = 1/y$. Therefore the posterior distribution has the form $p(N|y) = y/N^2$, $\quad N \geq y$. Therefore $p(N \leq n|y) = \int_y^n y/N^2 dN = y[-\frac{1}{n} + \frac{1}{y}] = 1 - y/n$. And so $p(N \leq n|y) = 1/2$ when $n = 2y$, therefore the limiting median in this example is 200.